Retrieving information from subordination

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Jean Bertoin[*] Marc Yor [†‡]

Abstract

We show that if $(X_s, s \geq 0)$ is a right-continuous process, $Y_t = \int_0^t \mathrm{d}s X_s$ its integral process and $\tau = (\tau_\ell, \ell \geq 0)$ a subordinator, then the time-changed process $(Y_{\tau_\ell}, \ell \geq 0)$ allows to retrieve the information about $(X_{\tau_\ell}, \ell \geq 0)$ when τ is stable, but not when τ is a gamma subordinator. This question has been motivated by a striking identity in law involving the Bessel clock taken at an independent inverse Gaussian variable.

Key words: Time-change, subordinator, information retrieval.

1 Introduction and main statements

1.1 Motivation

In Dufresne and Yor [4], it was remarked that by combining Bougerol's identity in law (see, e.g. Bougerol [3] and Alili *et al.* [1]) and the symmetry principle of Désiré André, there is the identity in distribution for every fixed $\ell \geq 0$

$$H_{\tau_\ell} \overset{\text{(law)}}{=} \tau_{a(\ell)}, \tag{1}$$

where $a(\ell) = \mathrm{Argsinh}(\ell) = \log\left(\ell + \sqrt{1+\ell^2}\right)$,

$$H_t = \int_0^t \mathrm{d}s R_s^{-2}, \qquad t \geq 0,$$

is the Bessel clock constructed from a two-dimensional Bessel process $(R_s, s \geq 0)$ issued from 1, and $(\tau_\ell, \ell \geq 0)$ is a stable $(1/2)$ subordinator independent from $(R_s, s \geq 0)$.

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In [4], the authors wondered whether (1) extends at the level of processes indexed by $\ell \geq 0$, or equivalently whether $(H_{\tau_\ell}, \ell \geq 0)$ has independent increments. Our main result entails that this is not the case. Indeed, Theorem 1 below implies that for every $\ell \geq 0$, the filtration $(\hat{\mathcal{H}}_\ell, \ell \geq 0)$ generated by $(H_{\tau_\ell}, \ell \geq 0)$ contains the filtration generated by $(R_{\tau_\ell}, \ell \geq 0)$. On the other hand, $((R_s, H_s), s \geq 0)$ is a Markov (additive) process, and since subordination by an independent stable subordinator preserves the Markov property, $((R_{\tau_\ell}, H_{\tau_\ell}), \ell \geq 0)$ is Markovian in its own filtration, which coincides with $\left(\hat{\mathcal{H}}_\ell, \ell \geq 0\right)$ by Theorem 1. It is immediately seen that for any $\ell' > 0$, the conditional distribution of $H_{\tau_{\ell+\ell'}}$ given $(R_{\tau_\ell}, H_{\tau_\ell})$ does not only depend on H_{τ_ℓ}, but on R_{τ_ℓ} as well. Consequently the process $(H_{\tau_\ell}, \ell \geq 0)$ is not even an autonomous Markov process. We point out that the process $(R_{\tau_\ell}, \ell \geq 0)$ is Markov (by subordination), and refer to a forthcoming paper [2] for details on the semigroup of $(R_{\tau_\ell}, H_{\tau_\ell})$.

1.2 Main results

More generally, we consider in this work an $\mathbb{R}^d$-valued process $(X_s, s \geq 0)$ with right-continuous sample paths, and $(\tau_\ell, \ell \geq 0)$ a stable subordinator with index $\alpha \in (0,1)$. We stress that we do *not require X and τ to be independent.* Introduce

$$Y_u = \int_0^u \mathrm{d}s X_s \,, \qquad u \geq 0 \,,$$

and the right-continuous time-changed processes

$$\hat{X}_\ell = X_{\tau_\ell} \text{ and } \hat{Y}_\ell = Y_{\tau_\ell}, \qquad \ell \geq 0 \,.$$

We are interested in comparing the information embedded in the processes $\hat{X}$ and $\hat{Y}$, respectively. We state our main result.

Theorem 1 *The right-continuous filtration $(\hat{\mathcal{Y}}_\ell, \ell \geq 0)$ generated by the process $\left(\hat{Y}_\ell, \ell \geq 0\right)$ contains the right-continuous filtration $(\hat{\mathcal{X}}_\ell, \ell \geq 0)$ generated by $\left(\hat{X}_\ell, \ell \geq 0\right)$.*

A perusal of the proof (given below in Section 2) shows that Theorem 1 can be extended to the case when it is only assumed that τ is a subordinator such that the tail of its Lévy measure is regularly varying at 0 with index $-\alpha$, which suggests that this result might hold for more general subordinators. On the other hand, if $(N_\ell, \ell \geq 0)$ is any increasing step-process issued from 0, such as for instance a Poisson process, then the time-changed process $(Y_{N_\ell}, \ell \geq 0)$ stays at 0 until the first jump time of N which is strictly positive a.s. This readily implies that the germ-σ-field

$$\bigcap_{\ell > 0} \sigma(Y_{N_\lambda}, \lambda \leq \ell)$$

is trivial, in the sense that every event of this field has probability either 0 or 1. Focussing on subordinators with infinite activity, it is interesting to point out that Theorem 1 fails when one replaces the stable subordinator τ by a gamma subordinator, as it can be seen from the following observation.

Proposition 1 *Let $\gamma = (\gamma_t, t \geq 0)$ be a gamma-subordinator and ξ a random variable with values in $(0,\infty)$ which is independent of γ. Then the germ-σ-field*

$$\bigcap_{t>0} \sigma(\xi\gamma_s, s \leq t)$$

is trivial.

We point out that Proposition 1 holds more generally when γ is replaced by a subordinator with logarithmic singularity, also called of class $(\mathcal{L})$, in the sense that the drift coefficient is zero and the Lévy measure is absolutely continuous with density g such that $g(x) = g_0 x^{-1} + G(x)$ where g_0 is some strictly positive constant and $G : (0,\infty) \to \mathbb{R}$ a measurable function such that

$$\int_0^1 |G(x)|\mathrm{d}x < \infty \ , \ g(x) \geq 0 \ , \ \text{and} \ \int_1^\infty g(x)\mathrm{d}x < \infty \, .$$

Indeed, it has been shown by von Renesse *et al.* [5] that such subordinators enjoy a quasi-invariance property analogous to that of the gamma subordinator, and this is the key to Proposition 1.

It is natural to investigate a similar question in the framework of stochastic integration. For the sake of simplicity, we shall focus on the one-dimensional case. We thus consider a real valued Brownian motion $(B_t, t \geq 0)$ in some filtration$(\mathcal{F}_t, t \geq 0)$ and an $(\mathcal{F}_t)$-adapted continuous process $(X_t, t \geq 0)$, and consider the stochastic integral

$$I_t = \int_0^t X_s \mathrm{d}B_s \, , \qquad t \geq 0 \, .$$

We claim the following.

Proposition 2 *Fix $\eta > 0$ and assume that the sample paths of $(X_t, t \geq 0)$ are Hölder-continuous with exponent η a.s. Suppose also that $(\tau_\ell, \ell \geq 0)$ is a stable subordinator of index $\alpha \in (0,1)$, which is independent of $\mathcal{F}_\infty$. Then the right-continuous filtration $(\hat{\mathcal{I}}_\ell, \ell \geq 0)$ generated by the subordinate stochastic integral $\left(\hat{I}_\ell = I_{\tau_\ell}, \ell \geq 0\right)$ contains the right-continuous filtration generated by $(|X_{\tau_\ell}|, \ell \geq 0)$.*

The proofs of these statements are given in the next section.

2 Proofs

2.1 Proof of Theorem 1.

We first observe that the proof can be reduced to showing that the germ-σ-field $\hat{\mathcal{Y}}_0$ contains the σ-field generated by $\hat{X}_0 = X_0$. Indeed, let us take this for granted, fix $\ell > 0$ and define $\tau'_u = \tau_{\ell+u} - \tau_\ell$ and $X'_v = X_{v+\tau_\ell}$. Then τ' is again a stable(α) subordinator and X' a right-continuous process, and

$$\hat{Y}_{\ell+u} - \hat{Y}_\ell = \int_0^{\tau'_u} \mathrm{d}v X'_v \,.$$

Hence $X'_0 = \hat{X}_\ell$ is measurable with respect to $\hat{\mathcal{Y}}_\ell$, and our claim follows.

Thus we only need to verify that X_0 is $\hat{\mathcal{Y}}_0$-measurable. In this direction, we shall use the following version of the Law of Large Numbers for the jumps $\Delta\tau_s = \tau_s - \tau_{s-}$ of a stable subordinator. Fix any $m > 2/\alpha$ and introduce for any given $b \in \mathbb{R}$ and $\varepsilon > 0$

$$N_{\varepsilon,b} = \mathrm{Card}\{s \le \varepsilon : b\Delta\tau_s > \varepsilon^m\} \,.$$

Note that $N_{\varepsilon,b} \equiv 0$ for $b \le 0$. For the sake of simplicity, we henceforth suppose that the tail of the Lévy measure of τ is $x \mapsto x^{-\alpha}$, which induces no loss of generality. So for $b > 0$, $N_{\varepsilon,b}$ is a Poisson variable with parameter

$$\varepsilon(\varepsilon^m/b)^{-\alpha} = b^\alpha \varepsilon^{1-m\alpha} \,.$$

Combining a standard argument based on the Borel-Cantelli lemma and Chebychev's inequality with monotonicity, we get that for $\varepsilon = 1/n$

$$\lim_{n\to\infty} n^{1-\alpha m} N_{1/n,b} = b^\alpha \qquad \text{for all } b > 0\text{, almost-surely.} \tag{2}$$

Let us assume that the process X is real-valued as the case of higher dimensions will then follow by considering coordinates. Set

$$J_\varepsilon = \mathrm{Card}\{s \le \varepsilon : \Delta\hat{Y}_s > \varepsilon^m\} \,,$$

where as usual $\Delta\hat{Y}_s = \hat{Y}_s - \hat{Y}_{s-}$. We note that

$$\Delta\hat{Y}_s - X_0\Delta\tau_s = \int_{\tau_{s-}}^{\tau_s} \mathrm{d}u(X_u - X_0) \,.$$

Hence if we set $a_\varepsilon = \sup_{0\le u\le \tau_\varepsilon} |X_u - X_0|$, then

$$(X_0 - a_\varepsilon)\,\Delta\tau_s \le \Delta\hat{Y}_s \le (X_0 + a_\varepsilon)\,\Delta\tau_s\,,$$

from which we deduce

$$N_{\varepsilon,X_0-a_\varepsilon} \le J_\varepsilon \le N_{\varepsilon,X_0+a_\varepsilon}\,.$$

Since X has right-continuous sample paths a.s., we have $\lim_{\varepsilon\to 0} a_\varepsilon = 0$ a.s., and taking $\varepsilon = 1/n$, we now deduce from (2) that

$$\lim_{n\to\infty} n^{1-m\alpha} J_\varepsilon = (X_0^+)^\alpha\,.$$

Hence X_0^+ is $\hat{\mathcal{Y}}_0$-measurable, and the same argument also shows that X_0^- is $\hat{\mathcal{Y}}_0$-measurable. This completes the proof of our claim. □

2.2 Proof of Proposition 1.

Let Ω denote the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R}_+$ endowed with the right-continuous filtration $(\mathcal{A}_t, t\ge 0)$ generated by the canonical process $\omega_t = \omega(t)$, and write $\mathbb{Q}$ for the law on Ω of the process $(\xi\gamma_t, t\ge 0)$.

It is well known that for every $x>0$ and $t>0$, the distribution of the process $(x\gamma_s, 0\le s\le t)$ is absolutely continuous with respect to that of the gamma process $(\gamma_s, 0\le s\le t)$ with density

$$x^{-t}\exp\left((1-1/x)\gamma_t\right)\,.$$

Because ξ and γ are independent, this implies that for any event $\Lambda\in\mathcal{A}_r$ with $r<t$

$$\mathbb{Q}\left(\Lambda\right) = \mathbb{E}\left(\xi^{-t}\exp\left((1-1/\xi)\gamma_t\right)\,\mathbf{1}_{\{\gamma\in\Lambda\}}\right)\,.$$

Observe that

$$\lim_{t\to 0+} \xi^{-t}\exp\left((1-1/\xi)\gamma_t\right) = 1 \qquad \text{a.s.}$$

and the convergence also holds in $L^1(\mathbb{P})$ by an application of Scheffé's lemma (alternatively, one may also invoke the convergence of backwards martingales). We deduce that for every $\Lambda\in\mathcal{F}_0$, we have

$$\mathbb{Q}(\Lambda) = \mathbb{P}(\gamma\in\Lambda)$$

and the right-hand-side must be 0 or 1 because the gamma process fulfills the Blumenthal's 0-1 law. □

2.3 Proof of Proposition 2.

The guiding line is similar to that of the proof of Theorem 1. In particular it suffices to verify that $|X_0|$ is measurable with respect to the germ-σ-field $\hat{\mathcal{I}}_0$.

Because B and τ are independent, the subordinate Brownian motion $(\hat{B}_\ell = B_{\tau_\ell}, \ell \geq 0)$ is a symmetric stable Lévy process with index 2α. With no loss of generality, we may suppose that the tail of its Lévy measure Π is given by $\Pi(\mathbb{R}\backslash[-x,x]) = x^{-2\alpha}$. As a consequence, for every $m > 2/\alpha$ and $\varepsilon > 0$ and $b \in \mathbb{R}$, if define

$$N_{\varepsilon,b} = \mathrm{Card}\{s \leq \varepsilon : |b\Delta\hat{B}_s|^2 > \varepsilon^m\},$$

then $N_{\varepsilon,b}$ is a Poisson variable with parameter $|b|^{2\alpha}\varepsilon^{1-m\alpha}$, and this readily yields

$$\lim_{n\to\infty} n^{1-\alpha m} N_{1/n,b} = |b|^{2\alpha} \qquad \text{for all } b \in \mathbb{R}, \text{ almost-surely.} \tag{3}$$

Next set

$$J_\varepsilon = \mathrm{Card}\{s \leq \varepsilon : |\Delta\hat{I}_s|^2 > \varepsilon^m\},$$

where as usual $\hat{I}_s = I_{\tau_s}$, and observe that

$$\Delta\hat{I}_s = X_0\Delta\hat{B}_s + (X_{\tau_{s-}} - X_0)\Delta\hat{B}_s + \int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\,. \tag{4}$$

Recall the assumption that the paths of X are Hölder-continuous with exponent $\eta > 0$, so the $(\mathcal{F}_t)$-stopping time

$$T = \inf\left\{u > 0 : \sup_{0\leq v<u} (u-v)^{-\eta}|X_u - X_v|^2 > 1\right\}$$

is strictly positive a.s. In particular, if we write $\Lambda_\varepsilon = \{\tau_\varepsilon < T\}$, then $\mathbb{P}(\Lambda_\varepsilon)$ tends to 1 as $\varepsilon \to 0+$.

We fix $a > 0$, consider

$$K_{\varepsilon,a} = \mathrm{Card}\left\{s \leq \varepsilon : \left|\int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\right|^2 > a\varepsilon^m\right\},$$

and claim that

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha m-1}\mathbb{E}(K_{\varepsilon,a}, \Lambda_\varepsilon) = 0\,. \tag{5}$$

If we take (5) for granted, then we can complete the proof by an easy adaptation of the argument in Theorem 1. Indeed, we can then find a strictly increasing sequence of integers $(n(k), k \in \mathbb{N})$

such that with probability one, for all rational numbers $a > 0$

$$\lim_{k\to\infty} n(k)^{1-\alpha m} K_{1/n(k),a} = 0\,. \tag{6}$$

We observe from (4) that for any $a \in (0, 1/2)$, if $|\Delta \hat{I}_s|^2 > \varepsilon^m$, then necessarily either

$$|X_0 \Delta \hat{B}_s|^2 > (1-2a)^2 \varepsilon^m\,,$$

or

$$|(X_{\tau_{s-}} - X_0)\Delta \hat{B}_s|^2 > a^2 \varepsilon^m\,,$$

or

$$\left| \int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}}) \mathrm{d}B_u \right|^2 > a^2 \varepsilon^m\,.$$

As

$$\lim_{\varepsilon\to 0+} \sup_{0\le s\le \varepsilon} |X_{\tau_{s-}} - X_0| = 0\,,$$

this easily entails, using (3) and (6), that

$$\begin{aligned} \limsup_{k\to\infty} n(k)^{1-\alpha m} J_{1/n(k)} &\le \lim_{k\to\infty} n(k)^{1-\alpha m} N_{1/n(k),(1-2a)^{-1}|X_0|} \\ &= (1-2a)^{-2\alpha} |X_0|^{2\alpha}\,, \qquad \text{a.s.} \end{aligned}$$

where the identity in the second line stems from (3). A similar argument also gives

$$\liminf_{k\to\infty} n(k)^{1-\alpha m} J_{1/n(k)} \ge (1+2a)^{-2\alpha} |X_0|^{2\alpha}\,, \qquad \text{a.s.,}$$

and as a can be chosen arbitrarily close to 0, we conclude that

$$\lim_{k\to\infty} n(k)^{1-\alpha m} J_{1/n(k)} = |X_0|^{2\alpha}\,, \qquad \text{a.s.}$$

Hence $|X_0|$ is $\hat{\mathcal{I}}_0$-measurable.

Thus we need to establish (5). As τ is independent of $\mathcal{F}_\infty$, we have by an application of Markov's inequality that for every $s \le \varepsilon$

$$\begin{aligned} &\mathbb{P}\left(\left| \int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}}) \mathrm{d}B_u \right|^2 > a\varepsilon^m, \Lambda_\varepsilon \mid \tau \right) \\ &\le \frac{1}{a\varepsilon^m} \int_0^{\Delta\tau_s} \mathrm{d}v v^\eta \;\le\; \frac{(\Delta\tau_s)^{1+\eta}}{a\varepsilon^m}\,. \end{aligned}$$

It follows that

$$
\begin{aligned}
\mathbb{E}(K_{\varepsilon,a},\Lambda_\varepsilon) &\leq \mathbb{E}\left(\sum_{s\leq\varepsilon}\left(\frac{(\Delta\tau_s)^{1+\eta}}{a\varepsilon^m}\wedge 1\right)\right) \\
&= \varepsilon c\int_{(0,\infty)} \mathrm{d}x x^{-1-\alpha}\left(\frac{x^{1+\eta}}{a\varepsilon^m}\wedge 1\right) = O(\varepsilon^{1-\alpha m/(1+\eta)})\,,
\end{aligned}
$$

where for the second line we used the fact that the Lévy measure of τ is $cx^{-1-\alpha}\mathrm{d}x$ for some unimportant constant $c > 0$. This establishes (5) and hence completes the proof of our claim. $\square$

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